How to Divide Things Fairly

STEVEN J. BRAMS
Department of Politics
New York University
New York, NY 10012
USA
steven.brams@nyu.edu

D. MARC KILGOUR
Department of Mathematics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
CANADA
mkilgour@wlu.ca

CHRISTIAN KLAMLER
Institute of Public Economics
University of Graz
A-8010 Graz
AUSTRIA
christian.klamler@uni-graz.at

Dividing a set of indivisible items, such as the marital property in a divorce, between two people can be a tricky business when the husband and wife rank the disputed property the same or similarly. But problems can arise even when they rank each item differently, as in this example:

$$\textit{Husband}: \text{ Sports Car} > \text{SUV} > \text{Boat} > \text{Desk} > \text{Couch} > \text{Painting}$$

$$\textit{Wife}: \text{ SUV} > \text{Boat} > \text{Sports Car} > \text{Couch} > \text{Painting} > \text{Desk}$$

At the outset, things look easy. Because we can give both spouses their first choices, it seems evident we should do exactly that, awarding the sports car to the husband and the SUV to the wife. It also seems clear that we should avoid giving them their last choices, which we can accomplish by awarding the painting to the wife and the desk to the husband.

But that leaves the boat and the couch, and we now have a problem: Both spouses prefer the boat to the couch, so who should get the boat? (Both also prefer the couch to the painting, but precluding the spouses' worst choices took care of this problem: The painting went to the wife, so the couch could go to the husband.)

An alternative approach to dividing marital property would be for the husband and wife to apply an algorithm, such as *alternation*, whereby the spouses take turns, each choosing one item when it is his or her turn. If the spouses are *sincere*—choose in the order of their rankings—and the husband starts, he gets {sports car, boat, desk}, and the wife gets {SUV, couch, painting}. If the wife starts, she does better, obtaining {SUV, boat, couch}, and the husband does worse, obtaining {sports car, desk, painting}. In the latter case, the husband does particularly badly, getting stuck with his worst item (the painting).

The items to be allocated need not be physical goods. For example, they could be committee assignments or project tasks, in which it is stipulated that each person is required to have three. Or they could be chores ("bads" instead of goods), in which case we could ask each person to rank them from least to most burdensome.

Math. Mag. **88** (2015) 338–348. doi:10.4169/math.mag.88.5.338.
MSC: Primary 91B32.

Can we go beyond the ad hoc criteria that we began with, such as giving the *players*—who need not be people but could be larger entities, such as organizations or nation-states—their best items and not their worst? This may not always be possible. If we use an algorithm, is there one that avoids the first-chooser bias of alternation, and is it applicable to more than two players?

Properties of Fair Division

The fair division of items, especially if they are indivisible or cannot be shared, is an age-old problem. In this article we describe a simple sequential algorithm, called SA, which seems to have been overlooked in earlier studies [**2, 3**], for carrying out this division when the players strictly rank items from best to worst. It is less demanding in the information it elicits than are algorithms that ask players to indicate their utilities for items [**10**], to rank bundles of items [**11**], or to apply the classic procedure of "divide-and-choose" [**9, 10**].

We begin by specifying the properties of an allocation for two or more players that it would be desirable to satisfy. Although SA uses only players' rankings, one of the four properties we describe below uses the *Borda score* of an item as one measure of its utility to a player: A lowest-ranked item receives 0 points, the next-lowest 1 point, and so on. A player's Borda score is the sum of its points for the subset of items it receives, which may be thought of as one possible cardinalization (into utilities) of the ranks. In the absence of the players' actual utilities for the items, which we assume to be additive, it is the only one we use in the subsequent analysis.

The properties of allocations that we analyze are the following, whose two-letter abbreviations we also use as adjectives in describing allocations:

- *Efficiency or Pareto-Optimality (PO)*: There is no other allocation that is at least as preferred by all players and strictly preferred by at least one.
- *Envy-freeness (EF)*: Each player values the set of items it receives at least as much as the set of items received by any other player.
- *Maximinality (MX)*: The allocation maximizes the minimum rank of the items received by any player.
- *Borda Maximinality (BMX)*: The allocation maximizes the minimum Borda score of the items received by any player.

MX ensures that the rank of the least-preferred item that any player receives is as high as possible [**7, 12**], whereas BMX ensures that the Borda score of the player with the lowest score is as high as possible [**8**]. As we will show, different allocations may satisfy each of these properties.

Because SA requires only that players rank items, we need a definition of envy-freeness that enables players to compare the value of their items with the value of the items received by the other players. We say that a player, say A, does not envy another player, say B, if and only if there is an injection (a 1-1 mapping) from A's items into B's items such that A prefers each of its items to the item of B to which it is mapped [**8**]. An allocation is *item-wise envy-free* (EF) if and only if no player envies any other.

To illustrate this definition in the two-person example we discussed in the introduction, assume that the husband receives {sports car, boat, desk} and the wife receives {SUV, couch, painting}. Then we can map item-wise the husband's items into the wife's such that he prefers each of his items to his wife's:

$$\text{sports car} > \text{SUV};\ \text{boat} > \text{couch};\ \text{desk} > \text{painting}.$$

Although there is a mapping for the wife such that she prefers two of her items to two of her husband's,

$$\text{SUV} > \text{sports car}; \ \text{painting} > \text{desk},$$

it is not true that couch > boat for her. Indeed, no allocation of three items to each spouse makes possible a 1-1 mapping such that each spouse item-wise prefers each of his or her items to the items of the other spouse. Thus, this example does not admit an EF allocation, based on item-wise comparisons.

The Sequential Algorithm (SA) and Examples

SA works in stages. We illustrate it with four examples in this section and also discuss its properties. In the following section we will prove more general results.

We assume that there are $n \geqslant 2$ players and $m = kn$ distinct items to be allocated, where k is a positive integer. If m is not a multiple of n (e.g., if $n = 2$ and m is odd), the "extra" items might be distributed to the players at random—with a maximum of one to each player—after SA has been applied.

SA produces an *equal* allocation, in which each player receives the same number k of items. If the allocation is not equal, it is not possible to make item-by-item comparisons, which our definition of EF assumes. We recognize that unequal allocations may be envy-free—based on the utilities that players have for their subsets of items, compared with the utilities they attribute to the subsets of items of the other players—but we cannot make this comparison based only on players' ranks.

The allocation rules of SA, which give one item to each player on each round, are the following:

(i) On the first round, descend the ranks of the players, one rank at a time, stopping at the first rank at which each player can be given a different item (at or above this rank). This is the *stopping point* for that round; the rank reached is its *depth*, which is the same for each player. Assign one item to each player in all possible ways that are at or above this depth (there may be only one), which may give rise to one or more SA allocations.

(ii) On subsequent rounds, continue the descent, increasing the depth of the stopping point on each round. At each stopping point, assign items not yet allocated in all possible ways until all items are allocated.

(iii) At the completion of the descent, if SA gives more than one possible allocation, choose one that is efficient (PO) and, if possible, EF.

The process of descent is the same as that of "fallback bargaining" [**4**], but its purpose is the fair division of items, not reaching an outcome acceptable to some (e.g., a simple majority) or all of the players.

We next give examples that illustrate rules (i)–(iii) when $n = 2$; later we analyze an example in which $n = 3$. The players are $A, B, \ldots$, and the items they rank are $1, 2, \ldots$. Players rank items in descending order of preference.

Example 1:

$$A: \ \underline{1}\ 2\ \underline{3}\ 4$$

$$B: \ \underline{2}\ 3\ \underline{4}\ 1$$

The stopping point of round 1 is depth 1, where A obtains item 1 and B obtains item 2. At depth 2 we cannot give different items to the players, because item 2 has already been given to B, so in round 2 we must descend to depth 3 to give the players different items (item 3 to A and item 4 to B).

We have underscored the items that each player receives. Because this exhausts the items, we are done, which yields the unique SA allocation of (13, 24) to (A, B). Henceforth, we list the players in alphabetical order, and their items in the order in which the players rank them.

Observe that on each round, each player prefers the item it receives to the item that the other player receives (for A, item 1 > item 2 and item 3 > item 4; for B, item 2 > item 3 and item 4 > item 1). Hence, there is a 1-1 mapping of A's items into B's, and B's items into A's, such that each player prefers its items to the other player's items. Therefore, the allocation (13, 24) is EF.

This allocation does *not* depend on a player's utilities for items, which we assume are consistent with their rankings (i.e., higher-ranked items have greater utility than lower-ranked items) and additive. Other two-item allocations, such as (12, 34), are not item-wise EF, because there is no 1-1 mapping of B's items to A's such that B prefers each of its items to the items to which it is mapped. In particular, notice in B's ranking that items 2 and 1 bracket items 3 and 4, so B may prefer the combination of items 2 (best) and 1 (worst) to the combination of items 3 and 4 (two middle-ranked items).

For example, if B's utilities for items 1, 2, 3, and 4 are 1, 5, 3, and 2, then B's utility for its subset of items, 34, is 5, and its utility for A's subset of items, 12, is 6, so B will envy A. But if B's utilities are 1, 6, 5, and 4, then it values its subset at 9 and A's subset at 7, so in this case B will not envy A. Only allocation (13, 24) is EF for all possible utilities of the players consistent with their rankings.

It is easy to see that (13, 24) is PO, because there is no allocation that is at least as preferred by both players. We say that (13, 24) is *Pareto-superior to*, or *Pareto-dominates*, another allocation—say, the "reverse" allocation, (24, 13)—because 1 > 2 and 3 > 4 for A, and 2 > 3 and 4 > 1 for B. In Example 1 no allocation is Pareto-superior to (13, 24), which means that (13, 24) is PO.

In general, an allocation is PO if and only if it is the product of a *sequence of sincere choices* by the players [**8**], whereby each player chooses its best available item on its turn. Thus, if the players choose items in the order *ABAB*, they obtain (13, 24); if they choose in the order *AABB*, they obtain (12,34), so both these allocations are PO. By comparison, no sincere sequence yields the allocation (24, 31), so it is not PO.

The allocation (13, 24) is also MX; the only other allocation of two items to each player that gives neither player a worst item is (12, 34), rendering it also MX. However, (12, 34) is *not* BMX, because it gives Borda scores of (5, 3) to (A, B), making B's score less than the score of 4 that each player receives from (13, 24).

If SA gives two or more allocations, only one may be MX. This is true of the two SA allocations—one on the left, the other on the right—in our next example (the vertical lines are explained below):

Example 2:

$$A:\ \underline{1}\,\underline{2}\,3\,4\,\underline{5}\mid 6\,\underline{7}\,8 \qquad A:\ \underline{1}\,\underline{2}\,3\,4\,\underline{5}\mid \underline{6}\,7\,8$$
$$B:\ \underline{3}\,\underline{4}\,5\,\underline{6}\,7\mid \underline{8}\,1\,2 \qquad B:\ \underline{3}\,\underline{4}\,5\,6\,\underline{7}\mid \underline{8}\,1\,2$$

In the first two rounds, SA gives (12, 34) to (A, B), reaching depth 2 in both allocations. In round 3, the stopping point is depth 5 for both the left and right allocations, as shown by the vertical lines, but now there is some choice in the items we give to A and B. In particular,

(i) the left-hand allocation gives items (5, 6) to (A, B) at depth 5, followed in round 4 by items (7, 8) at depth 7, resulting in (1257, 3468);

(ii) the right-hand allocation gives items (5, 7) to (A, B) at depth 5, followed in round 4 by items (6, 8) at depth 6, resulting in (1256, 3478).

Clearly the right-hand allocation, with a maximum depth of 6, is MX. Note that all allocations must have depth 6 or greater; otherwise, item 8 would not be assigned to either player.

The right-hand allocation (1256, 3478) is also BMX, giving (A, B) Borda scores of (18, 18)—a minimum score of 18—whereas the left-hand allocation (1257, 3468) gives the players scores of (17, 19) for a minimum score of 17. (An exhaustive search shows that no other allocation, even among allocations in which players receive different numbers of items, gives a greater minimum than 18.) Thus, while both allocations are EF and PO, only the right-hand allocation is MX and BMX.

Example 2 illustrates how SA can result in more than one allocation that is PO and EF, but only one is MX or BMX in this example. Our next example illustrates that if both players rank the same item last, there cannot be an EF allocation:

Example 3:

$$A: \underline{1}\ 2\ \underline{3}\ \underline{4}\ 5\ 6 \qquad A: \underline{1}\ 2\ \underline{3}\ 4\ 5\ \underline{6}$$
$$B: \underline{2}\ 3\ \underline{5}\ 4\ 1\ \underline{6} \qquad B: \underline{2}\ 3\ \underline{5}\ \underline{4}\ 1\ 6$$

In rounds 1 and 2, with stopping points at depth 1 and depth 3, the left-hand and right-hand allocations coincide, giving 13 to A and 25 to B. In round 3, the stopping point is depth 6 for both allocations, the lowest possible, but items 4 and 6 are assigned in two different ways.

The player who receives item 6 must be envious, because no 1-1 mapping can map item 6 to a less-preferred item. Thus, the allocation of items in Example 3 is not EF, although the partial allocation of the first four items to both players at depths 1 and 3 is. Both complete allocations are MX (maximum depth of 6) and BMX (minimum Borda score of 8).

Our final example in this section, which duplicates the rankings of the husband (A) and wife (B) in the introduction, illustrates that an SA allocation may fail to be EF even when the players rank every item differently (we illustrated this failure earlier for alternation but not for SA):

Example 4:

$$A: \underline{1}\ 2\ 3\ \underline{4}\ \underline{5}\ 6 \qquad A: \underline{1}\ 2\ \underline{3}\ \underline{4}\ 5\ 6$$
$$B: \underline{2}\ \underline{3}\ 1\ 5\ \underline{6}\ 4 \qquad B: \underline{2}\ 3\ 1\ \underline{5}\ \underline{6}\ 4$$

Here the problem arises in round 2, with the stopping point at depth 4, where the left-hand allocation gives 14 to A and 23 to B, whereas the right-hand allocation gives 13 to A and 25 to B. Whoever receives the lower-ranked item in round 2 will be envious, because no 1-1 matching can map every item of that player to a lower-ranked item of the other player. Despite the fact that each player ranks the six items differently, and no player receives a worst item in either SA allocation, neither allocation in Example 4 is EF.

In Example 4, both allocations are MX (in the descent, the depth reaches 5 when the allocations are complete); moreover, they are equal according to BMX, with Borda scores of 10 to the advantaged player and 8 to the disadvantaged player, so the minimum Borda score of each allocation is 8. According to both MX and BMX, therefore, the two allocations in Example 4, unlike Example 3, are equally fair to the players.

When $n = 2$, there is a simple condition, called "Condition D" [**6**], for determining whether an EF allocation can exist: For every odd i, the two players' sets of top i items are *not* identical. In Example 3, A's and B's top 5 items are identical, and in Example 4, their top 3 items are identical, so neither example yields an EF allocation. This is not true for the top 1 and top 3 items in Example 1, nor the top 1, 3, 5, and 7 items in Example 2, so in both of these examples an EF allocation exists which, as we showed, SA finds.

To summarize, SA may give a unique PO-EF allocation (Example 1) or multiple PO-EF allocations (Example 2), only one of which—possibly not the same one—is MX or BMX. In addition, SA may not produce an EF allocation (Examples 3 and 4), even when the players rank all items differently (Example 4), because no EF allocation exists.

Properties of SA

If all items can be allocated in an EF way, we say there is a *complete EF allocation*. For $n = 2$, Brams, Kilgour, and Klamler [**6**] provide an algorithm, AL (for "algorithm"), which finds at least one complete PO-EF allocation if one exists (though not necessarily all of them, as we will see). Furthermore, when there is no complete EF allocation, as in Examples 3 and 4, AL finds the largest and most preferred subset of items that can be allocated in an EF way. Items that cannot be so allocated (e.g., items 4 and 6 in Example 3; items 3 and 6 in Example 4) are placed in a "contested pile," to which another algorithm, called undercut, can be applied [**1, 5**].

By contrast, SA always allocates all items. As illustrated in Examples 3 and 4, SA yields an allocation that may be EF only on some rounds, rendering the allocation only partially EF.

Although SA may not give a complete EF allocation, it always produces at least one PO allocation. Moreover, this is true however many players there are (i.e., for all $n \geqslant 2$).

Theorem 1. *SA rules (i) and (ii) produce at least one allocation that is PO.*

Proof. Under SA, all items are allocated one at a time to the players and ranked at or above each stopping point in the descent process. Because each allocation gives equal numbers of items to the players, a non-SA allocation must give at least one player an item it ranks below some item that it would receive under SA. This proves that no non-SA allocation can be Pareto-superior to any SA allocation. Because Pareto-superiority is irreflexive and transitive, at least one of the SA allocations—say, X—must be maximal with respect to Pareto-superiority within the set of SA allocations. Because no non-SA allocation can be Pareto-superior to X, X must be PO. ∎

Theorem 1 guarantees that at least one PO allocation survives the application of rule (iii). Our next example shows that rule (iii) has bite—not every SA allocation need be PO. This example also shows that not all allocations given by AL, which duplicate those given by SA in Example 5, need be "locally Pareto-optimal," as we incorrectly claimed in [**6**].

Example 5:

$$A: \underline{1}\ \underline{2}\ 3\ \underline{4}\ 5 \mid \underline{6}\ 7\ 8 \qquad A: \underline{1}\ \underline{2}\ 3\ 4\ \underline{5} \mid \underline{6}\ 7\ 8$$

$$B: \underline{7}\ \underline{8}\ 1\ 2\ \underline{3} \mid \underline{5}\ 4\ 6 \qquad B: \underline{7}\ \underline{8}\ 1\ 2\ \underline{3} \mid 5\ \underline{4}\ 6$$

At the completion of round 2, SA gives (12, 78) to (A, B), stopping at depth 2. The next stopping point is at depth 5, indicated by the vertical lines, where B must receive item 3 in round 3. There is, however, a choice for A: The left-hand allocation gives item 4 to A, and the right-hand allocation gives item 5 to A.

Continuing, the left-hand allocation gives items 6 and 5, respectively, to A and B in round 4, with the stopping point at depth 6, whereas the right-hand allocation gives items 6 and 4 to A and B, respectively, in round 4, with the stopping point at depth 7. Because both players prefer the last two items they receive in the left-hand allocation to those that they receive in the right-hand allocation, the left-hand allocation Pareto-dominates the right-hand allocation, so only the left-hand allocation is PO.

Interestingly enough, both SA allocations in Example 5 are complete EF allocations, even though only one is PO, showing that EF does not imply PO. The converse also fails because, for example, an allocation that gives one player only its top items will generally make another player envious. Thus, PO and EF are independent properties.

In Example 5, the left-hand allocation (1246, 7835) is MX (its maximum depth is 6). An exhaustive search of equal and unequal allocations shows that it is also BMX, with Borda scores of (19, 18), compared with Borda scores of (18, 17) for the right-hand allocation.

But SA does not invariably find a PO-EF allocation that—based on the properties of MX or BMX—is superior to a non-SA allocation, as our next example illustrates.

Example 6:

$$\begin{array}{llll} A: & \underline{1}\,\underline{2}\,\underline{3}\,\underline{4}\,5\,6\,7\,8 & \qquad A: & \underline{1}\,\underline{2}\,3\,\underline{4}\,\underline{5}\,6\,7\,8 \\ B: & \underline{8}\,\underline{7}\,\underline{6}\,3\,2\,1\,\underline{5}\,4 & \qquad B: & \underline{8}\,\underline{7}\,\underline{6}\,\underline{3}\,2\,1\,5\,4 \end{array}$$

The SA allocation is shown on the left. In the first three rounds, at depths 1, 2, and 3, SA allocates (123, 876) to (A, B). On round 4 and at depth 7, A and B receive, respectively, items 4 and 5, producing the allocation (1234, 8765).

But the non-SA allocation (1245, 8763) on the right is of depth 5. Moreover, it is not only MX but also BMX, giving Borda scores of (20, 22), compared with (22, 19) for the SA allocation. Both the left-hand and the right-hand allocations are EF and PO. We will return to this example later to show that there are seven distinct complete EF allocations, but only the aforementioned two are PO.

Both the left-hand (SA) and the right-hand (non-SA) allocations in Example 6 are complete EF and PO (A prefers the former, and B the latter, when each player obtains its four best items). AL gives only the SA allocation, so like SA, it does not always find all PO-EF allocations—including those that might be MX or BMX (e.g., the non-SA allocation on the right in Example 6)—as we incorrectly stated in [**6**].

Although at least one SA allocation is PO by Theorem 1, it may not be EF, as we showed in Examples 3 and 4. But if there is an EF allocation when $n = 2$, we have the following result.

Theorem 2. *Let $n = 2$. If an EF allocation exists, then SA will give at least one allocation that is EF and PO.*

Proof. We earlier mentioned Condition D (see Example 4)—that an EF allocation exists if and only if, for all odd k, at least one of A's k most preferred items is not one of B's $k$most preferred items. Another necessary and sufficient condition for an allocation to be EF is that the item that each player receives on the jth round is among the player's top $2j - 1$ items [**6**].

Assume Condition D holds. Taking $k = 1$, it is clear that A's and B's most preferred items are different, so on round $j = 1$, SA must allocate to each player its most preferred item, and the stopping depth d_j is $d_1 = 1$.

Now assume that, up to the completion of round j, SA has allocated j of each player's top $2j - 1$ items, and the stopping depth on round j is $d_j \leqslant 2j - 1$. Consider round $j + 1$. Combined, the preference orderings of A and B account for either 2, 3, or 4 distinct additional items at depth $2j$ or $2j + 1$. Therefore, to assign an additional item to both A and B from their top $2j + 1$ items, it is necessary to increase the stopping depth to at most $d_j + 2$.

If there is a choice, ensure that a player does not prefer any unassigned item to the item it receives. It follows that $d_{j+1} \leqslant 2j + 1$, and that the $(j + 1)$st item received by each player is among its $2j + 1$ most preferred items. Therefore, the resulting SA allocation is EF. Moreover, it is PO, because it is the result of a sequence of sincere choices (as discussed after Example 1). ■

When $n = 2$, it is relatively easy to determine whether a given allocation is EF, PO, MX, or BMX. It is considerably more complex to find *all* allocations that are, say, EF.

To illustrate this calculation, recall from Example 6 that we gave a non-SA equal allocation that improved upon the SA allocation in terms of MX and BMX, but we did not prove that it was the only such allocation, or that there was not another allocation that better satisfied one or both of these properties. To analyze Example 6 in detail, we list all possible item-by-item allocations at each odd depth.

Example 6 (repeated):

$$A:\ 1\,2\,3\,4\,5\,6\,7\,8$$

$$B:\ 8\,7\,6\,3\,2\,1\,5\,4$$

At depth 1, (A, B) must receive items (1, 8). Then, at depth 3, (A, B) must receive, in addition, one of (2, 7), (2, 6), (3, 7), or (3, 6). Finally, at depth 5 and again at depth 7, (A, B) must receive pairs of items that depend on the items already received. The details are shown in Table 1, which includes all EF allocations for Example 6, as well as their MX depths and Borda scores, illustrating that the determination of all EF allocations and their properties may be combinatorially complex.

As Table 1 shows, there are seven EF allocations, labeled a, b, c, d, e, f, and g, which we call *classes*, that can be reached in a total of 21 different ways. Specifically, there are 7 a's, 4 b's, 2 c's, 1 d, 1 e, 5 f's, and 1 g. The MX depths and Borda scores depend only on the class, not on the way it was obtained. These scores are shown only for the first member of each class.

The MX depths of the b's and the f's are minimal (i.e., 5), but only the b's have a maximin Borda score (20). This verifies that allocation b (1245, 8765) is indeed MX and BMX. It and the unique SA allocation (allocation 1 in class a) are the only PO allocations.

So far we have not illustrated SA with examples in which $n > 2$. While its application to the division of items among three or more players is straightforward, if more tedious, SA no longer ensures that if there is a complete EF allocation, it will be chosen by SA when $n > 2$.

Example 7:

$$A:\ \underline{1}\,\underline{2}\,3\,4\,5\,\underline{6}\,7\,8\,9$$

$$B:\ \underline{5}\,\underline{8}\,1\,2\,6\,\underline{7}\,3\,4\,9$$

$$C:\ \underline{3}\,\underline{4}\,\underline{9}\,1\,2\,5\,6\,7\,8$$

SA allocates items (1, 5, 3) to (A, B, C) at depth 1; then (2, 8, 4) at depth 2; and finally (6, 7, 9) at depth 6. Notice that B may envy A for obtaining items {1, 2, 6}, which fall

TABLE 1: EF Allocations, MX Depths, and Borda Scores for Example 6

Allocation	**Depth $\leqslant$ 3**	**Depth $\leqslant$ 5**	**Depth $\leqslant$ 7**	**Complete**	**MX Depth**	**BMX Score**
1	(2, 7)	(3, 6)	(4, 5)	(1234, 8765)-*a*	7	(22, 19)
2		(4, 6)	(3, 5)	(1243, 8765)-*a*		
3			(5, 3)	(1245, 8763)-*b*	5	(20, 22)
4		(4, 3)	(5, 6)	(1245, 8736)-*b*		
5			(6, 5)	(1246, 8735)-*c*	7	(19, 18)
6		(5, 6)	(4, 3)	(1254, 8763)-*b*		
7		(6, 3)	(4, 5)	(1264, 8735)-*c*		
8	(2, 6)	(3, 5)	(4, 7)	(1234, 8657)-*a*		
9		(4, 3)	(5, 7)	(1245, 8637)-*b*		
10			(7, 5)	(1247, 8635)-*d*	7	(18, 17)
11	(3, 7)	(2, 6)	(4, 5)	(1324, 8765)-*a*		
12		(4, 6)	(2, 5)	(1342, 8765)-*a*		
13		(4, 2)	(6, 5)	(1346, 8725)-*e*	7	(18, 17)
14			(5, 2)	(1345, 8762)-*f*	5	(19, 21)
15		(5, 6)	(4, 2)	(1354, 8762)-*f*		
16		(5, 2)	(4, 6)	(1354, 8726)-*f*		
17	(3, 6)	(2, 7)	(4, 5)	(1324, 8675)-*a*		
18		(2, 5)	(4, 7)	(1324, 8657)-*a*		
19		(4, 2)	(5, 7)	(1345, 8627)-*f*		
20			(7, 5)	(1347, 8625)-*g*	7	(17, 16)
21		(5, 7)	(4, 2)	(1354, 8672)-*f*		

Note: At depth 1, the 21 complete EF allocations give items (1, 8) to (A, B). At lower depths, they fall into seven classes (7 *a*'s, 4 *b*'s, 2 *c*'s, 1 *d*, 1 *e*, 5 *f*'s, 1 *g*), each of which gives the same complete allocation but different items at different maximum odd depths. The MX depths, and the BMX scores, are shown only for the first member of each class. The MX depths of the *b*'s and the *f*'s are minimal (5), but only the *b*'s have a maximin Borda score (20). The *a*'s and the *b*'s are the only two classes that yield PO allocations, with the first *a* allocation (allocation 1) being the unique SA allocation.

between B's two best items (items 5 and 8) and its sixth-best item (item 7). Because A's items bracket B's, it follows that there is no 1-1 mapping of B's items to A's such that B always prefers its own item to the item of A to which it is mapped. Thus, this allocation is not EF.

However, by switching items 6 and 7 between A and B in the SA allocation, we obtain a non-SA allocation, as demonstrated below.

Example 7 (cont.):

$$A: \ \underline{1}\,\underline{2}\,3\,4\,5\,6\,\underline{7}\,8\,9$$

$$B: \ \underline{5}\,\underline{8}\,1\,2\,\underline{6}\,7\,3\,4\,9$$

$$C: \ \underline{3}\,\underline{4}\,\underline{9}\,1\,2\,5\,6\,7\,8$$

To show that the allocation (127, 586, 349) is EF, observe that C gets its three best items, so it cannot do better and, therefore, will not be envious. But now it is easy to check that the required 1-1 mappings of A's items to B's, B's to A's, and A's and B's to C's, all exist, confirming that the allocation is EF.

As illustrated in Example 1, we can similarly demonstrate that the SA allocation in Example 7 is PO with the sequence of sincere choices, *ABCABCCAB*. To demonstrate that the non-SA allocation is also PO, we can use the sequence of sincere choices, *ABCABCCBA*.

Although not EF, the SA allocation in Example 7 has the advantage of being both MX and BMX. It gives A and B at worst a sixth-best item, whereas the non-SA allocation gives A a seventh-best item. Similarly, the SA allocation gives Borda scores of (18, 18, 21) to (A, B, C), whereas the EF allocation gives the players Borda scores of (17, 19, 21), so the SA allocation gives a higher minimum. Clearly there are trade-offs among our properties, and which should take priority may be open to debate.

As a final property of SA, we consider its vulnerability of manipulation. Not surprisingly, if $n = 2$ and one player (say, A) has complete information about the preferences of the other player (B), and B is sincere, A can exploit B, as shown in our next example.

Example 8:

$$A: \ \underline{1}\,\underline{2}\,3\,\underline{4}\,5\,6$$

$$B: \ \underline{6}\,\underline{3}\,\underline{5}\,4\,2\,1$$

The SA allocation is underscored, with B receiving its three top items and A not doing quite so well. But now assume that A insincerely indicates its preferences to be those shown below, with B's preferences remaining the same.

Example 8 (cont.):

$$A': \ \underline{3}\,\underline{1}\,\underline{2}\,4\,5\,6$$

$$B: \ \underline{6}\,3\,\underline{5}\,\underline{4}\,2\,1$$

This SA allocation shows that A's insincere preferences turn its original disadvantage into an advantage by giving it its three top items, whereas B now does worse.

Although not strategy-proof, SA seems relatively invulnerable to strategizing in the absence of any player's having complete information about its opponent's or opponents' preferences. The manipulator's task is further complicated if the other players are aware that an opponent might try to capitalize on its information and, consequently, they take countermeasures (e.g., through deception) to try to prevent their exploitation.

Summary and Conclusions

To summarize, we have shown that if $n \geqslant 2$, SA always yields at least one PO allocation and, if $n = 2$, SA always yields an allocation that is PO and EF, provided an EF allocation exists. Although, initially, SA may produce some allocations that are not PO, these will be eliminated by invoking SA rule (iii). The set of PO-EF allocations that SA produces, however, may not include one that satisfies the properties of MX or BMX, although our examples suggest that it probably will not be far off.

If $n > 2$, SA may fail to yield an EF allocation when one exists. In such a case, however, an SA allocation may have redeeming properties, such as be MX or BMX. While SA is not strategy-proof even when $n = 2$, in most real-life cases it is unlikely that one player would have sufficient information about another player's preference rankings—not to mention be able to formulate a strategy that would exploit such information—to manipulate it successfully.

SA seems most applicable to allocation problems in which there are numerous small items, which need not be physical goods, as we noted earlier. If there is one big item that two players desire (e.g., the house in a divorce), it may not be possible to prevent envy, especially because SA specifies that each player must receive the same number of items. (This stipulation may be viewed as essential to achieving fairness in some, but certainly not all, situations.) In such a case, the most practical solution might be to sell the big item—in effect, making it divisible—and divide the proceeds.

Other modifications in SA might include not restricting the allocation of items of one to each player on every round, and relaxing the assumption that the number of items is an integer multiple of the number of players. These modifications would change the fair-division problem fundamentally, however, because properties like EF, MX, and BMX would have to be redefined to take into account that players may not receive the same number of items.

Acknowledgment The authors thank the editor and two referees for valuable suggestions that have substantially improved this article. Klamler was supported by the University of Graz project "Wahl-Auswahl-Entscheidung."

REFERENCES

1. H. Aziz, A note on the undercut procedure, *Social Choice Welfare* **45** (2015) 723–728.
2. S. Bouveret, Y. Chevaleyre, N. Maudet, Fair allocation of indivisible goods. In *Handbook of Computational Social Choice*. Ed. F. Brandt, V. Conitzer, U. Endriss, J. Lang, A. D. Procaccia. Cambridge Univ. Press, Cambridge, UK, 2016.
3. S. J. Brams, P. H. Edelman, P. C. Fishburn, Fair division of indivisible items, *Theory Decis.* **55** no. 2 (2003) 147–180.
4. S. J. Brams, D. M. Kilgour, Fallback bargaining, *Group Decis. Negotiation* **10** no. 4 (2001) 287–316.
5. S. J. Brams, D. M. Kilgour, C. Klamler, The undercut procedure: An algorithm for the envy-free division of indivisible items, *Social Choice Welfare* **39** nos. 2–3 (2012) 615–631.
6. S. J. Brams, D. M. Kilgour, C. Klamler, Two-person fair division of indivisible items: An efficient, envy-free algorithm, *Not. AMS* **61** no. 2 (2014) 130–141.
7. S. J. Brams, D. M. Kilgour, C. Klamler, Maximin envy-free division of indivisible items (to be published).
8. S. J. Brams, D. L. King, Efficient fair division: Help the worst off or avoid envy?, *Ration. Soc.* **17** no. 4 (2005) 387–421.
9. S. J. Brams, A. D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge Univ. Press, Cambridge, UK, 1996.
10. S. J. Brams, A. D. Taylor, *The Win-Win Solution: Guaranteeing Fair Shares to Everybody*. W. W. Norton, New York, 1999.
11. D. Herreiner, C. Puppe, A simple procedure for finding equitable allocations of indivisible goods, *Social Choice Welfare* **19** no. 2 (2002) 415–430.
12. A. D. Procaccia, J. Wang, Fair enough: Guaranteeing approximate maximin shares. *Proceedings of the 14th ACM Conference on Economics and Computation* (2014) 675–692.

Summary. We propose an intuitively simple sequential algorithm (SA) for the fair division of indivisible items that are strictly ranked by two or more players. We analyze several properties of the allocations that it yields and discuss SA's application to real-life problems, such as dividing the marital property in a divorce or assigning people to committees or projects.

STEVEN J. BRAMS (MR Author ID: 40915) has long been interested in fair division and is the author of three books on the subject, the latest being *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures* (Princeton University Press, 2008). He is Professor of Politics at New York University and works regularly with mathematicians, economists, and computer scientists on a variety of interdisciplinary topics.

D. MARC KILGOUR (MR Author ID: 101405) is Professor of Mathematics at Wilfrid Laurier University in Waterloo, Canada. Beginning from a base in game theory, he works at the intersection of mathematics, engineering, and social science and has contributed on topics ranging from environmental management and decision support to electoral systems and fair division.

CHRISTIAN KLAMLER (MR Author ID: 730171) is Associate Professor of Economics at the University of Graz in Austria. His research interests lie in the algorithmic and the normative analysis of fair-division procedures and voting rules.